**Application of integrals (examples)**

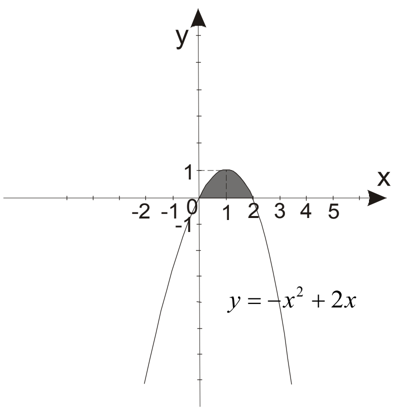
**1. Calculate the area of the figure limited with curve $y=-x^2+2x$ and line $y = 0$.**

***Solution:***

In these tasks we must first draw the picture and find the point of intersection because they give us borders of integral.

i) $y=-x^2+2x \longrightarrow -x^2+2x=0 \longrightarrow x = 0$ and $x = 2$

ii) $y' = -2x + 2$, $y' = 0$ for $-2x+2 = 0 \longrightarrow x = 1 \longrightarrow y = -1^2+2 = 1$, the point (1,1) is maximum.




We need to find this area , and it is clear that the limits of integrals go from 0 to 2, so:

$$A=\int_0^2(-x^2+2x)dx=\left(-\frac{x^3}{3}+2\frac{x^2}{2}\right)\Bigg|_0^2=[(-\frac{2^3}{3}+2^2)-(\frac{0}{3}+0)]=-\frac{8}{3}+4=\frac{4}{3}$$

**2. Calculate the area of the figure, which is limited with lines: $y=2x^2+1$ and $y=x^2+10$**

***Solution:***

Points of intersection of the two curves, we get as the solution of system equations( that will give us the border of integral):

$$y=2x^2+1$$
$$y=x^2+10$$

$2x^2 + 1 = x^2 + 10$

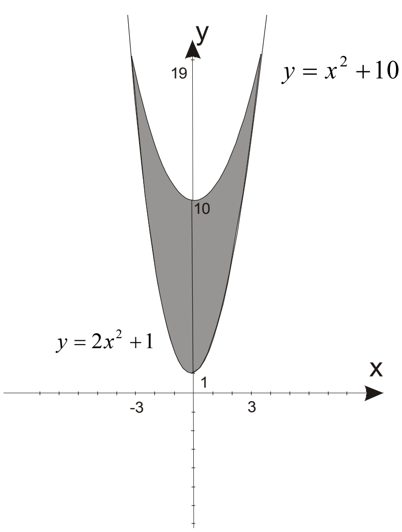
$x^2 = 9$ **So integral "goes" from -3 to 3**

$x = \pm 3$

**Next examine a few "things" to draw graphics:**

| | |
|---|---|
| $y = 2x^2 + 1$ | $y = x^2 + 10$ |
| $2x^2 + 1 = 0$ | $x^2 + 10 = 0$ |
| $x^2 = -\frac{1}{2}$ | $x^2 = -10$ |

| | |
|---|---|
| $y = 2x^2 + 1$ | $y = x^2 + 10$ |
| y` = 4x | y`= 2x |
| 4x = 0 | 2x = 0 |
| **x = 0** | **x = 0** |
| **y**= 1 | **y= 10** |
| (0,1) is minimum | (0,10) is minimum |




$$A = \int_{-3}^{3} [(x^2 + 10) - (2x^2 + 1)]dx$$

**Important: Since the graph is symmetrical in relation to the y-line, it is easier for us to calculate the area from 0 to 3 and to multiply that with 2** …

$$A = 2\int_0^3 [(x^2+10)-(2x^2+1)]dx = 2\int_0^3 (-x^2+9)dx = 2(-\frac{x^3}{3}+9x)\Big|_0^3 = 2*18 = 36$$

**3. Determine the area limited with $y^2+y+x=6$ and y – line.**

***Solution:***

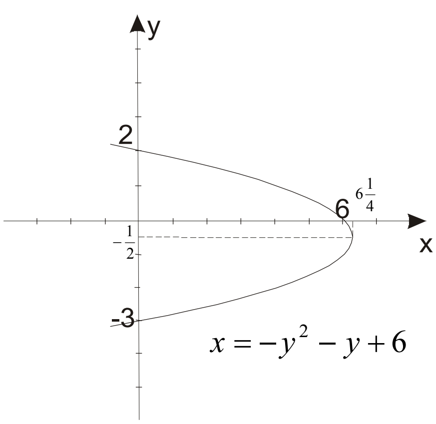
**In this task is smarter to express x, and to calculate the required area "by y"…**

$y^2+y+x=6$

$x=-y^2-y+6$ ⟶ $-y^2-y+6=0$ ⟶ $y_{1,2}=\frac{1\pm 5}{-2}$ ⟶ $y_1 = -3$, $y_2 = 2$

$x' = -2y-1$; So: $x' = 0$ for $-2y-1=0$ then is $y=-\frac{1}{2}$ and $x=6\frac{1}{4}$

Point **$(6\frac{1}{4}, -\frac{1}{2})$ is maximum when we think "by y"**

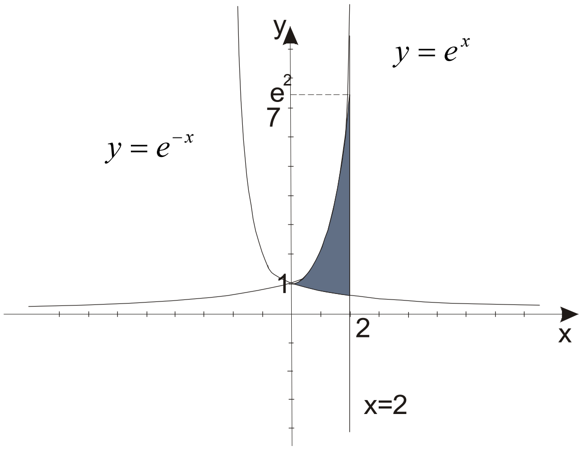



$$A = \int_{-3}^{2}(-y^2-y+6)dy = (-\frac{y^3}{3}-\frac{y^2}{2}+6y)\Big|_{-3}^{2} = \frac{125}{6}$$

**4. Calculate the area of the figure, which is limited with lines $y = e^x$ , $y = e^{-x}$ and $x = 2$**

***Solution:***

Here we have a graphics of basic functions. If you are not familiar with them, create a table of values…( choose values for x and then find y).




$$A = \int_0^2 (e^x - e^{-x})dx = (e^x + e^{-x})\Big|_0^2 = (e^2 + e^{-2}) - (e^0 + e^{-0}) = e^2 + e^{-2} - 2$$

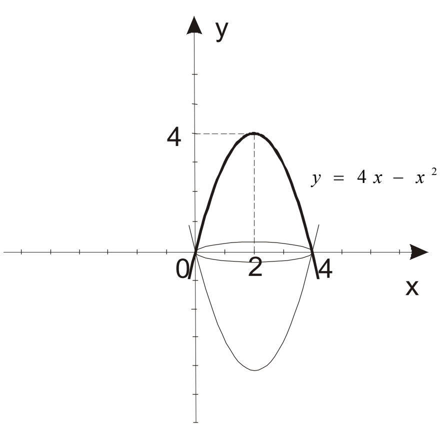
**5. Calculate volume of body which make parable $y = 4x - x^2$ when she rotates around x – line.**

***Solution:***

$y = 4x - x^2$

$4x - x^2 = 0 \longrightarrow x(4-x) = 0 \Rightarrow x = 0 \vee x = 4$

y\` = 4-2x $\longrightarrow$ 4-2x = 0 $\longrightarrow$ x = 2 $\longrightarrow$ y = 4




Borders are 0 and 4

$$V = \pi\int_a^b y^2 dx$$

$$V = \pi\int_0^4 (4x - x^2)^2 dx = \pi\int_0^4 (16x^2 - 8x^3 + x^4)dx$$

$$= \pi(16\frac{x^3}{3} - 8\frac{x^4}{4} + \frac{x^5}{5})\Big|_0^4$$

$$= \pi(16\frac{64}{3} - 2\circ 256 + \frac{256}{5}) = \pi\frac{512}{15} = \frac{512\pi}{15}$$

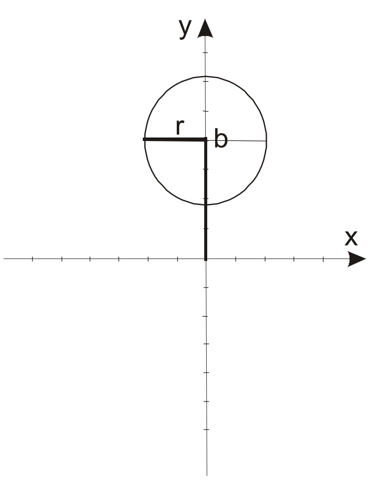
**Volume is** $\frac{512\pi}{15}$

**6. Find volume of body which caused circle** $x^2 + (y-b)^2 = r^2$ **rotating around x – line (b>r)**

***Solution:***

From the analytical geometry we know that the equation of circle is $(x-p)^2 + (y-q)^2 = r^2$ where are **p** and **q** center coordinates, and **r**- radius of circle.

$x^2 + (y-b)^2 = r^2$ ⟶ p = 0 and q =b, so:

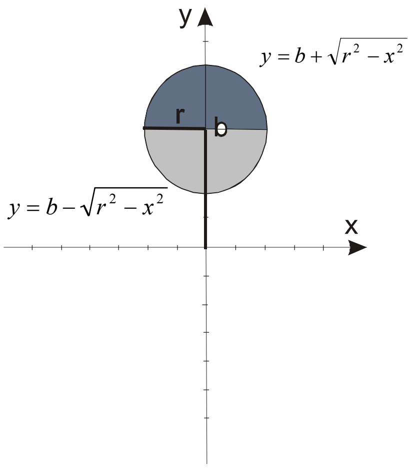



$x^2 + (y-b)^2 = r^2$ **here we have to expres y**

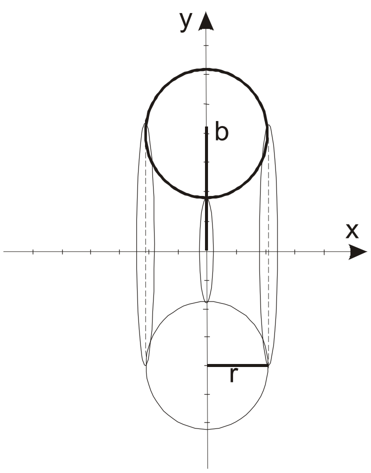
$(y-b)^2 = r^2 - x^2$

$y - b = \pm\sqrt{(r^2 - x^2)}$

$y = b \pm \sqrt{(r^2 - x^2)}$ Here we get two circles: the upper $y = b + \sqrt{(r^2 - x^2)}$ and lower $y = b - \sqrt{(r^2 - x^2)}$




Rotation of this circle will give us the body, which is known as **TORUS.**




V = $\pi\int_a^b (y_1^2 - y_2^2)dx$

Find first value $y_1^2 - y_2^2$

$y_1^2 - y_2^2 = (b + \sqrt{(r^2 - x^2)}\,)^2$ - $(b - \sqrt{(r^2 - x^2)})^2$

$= (b^2 + 2b\sqrt{r^2 - x^2} + r^2)$ - $(b^2 - 2b\sqrt{r^2 - x^2} + r^2)$

$= b^2 + 2b\sqrt{r^2 - x^2} + r^2$ **-** $b^2 + 2b\sqrt{r^2 - x^2} - r^2$

$= \quad 4b\sqrt{r^2 - x^2}$

**It is clear that boundaries are – r and r**

**First, we will solve integral:**

$$\int\sqrt{r^2-x^2}dx = \begin{vmatrix} x = r\sin t \\ dx = r\cos t dt \end{vmatrix} = \int\sqrt{(r^2-r^2\sin^2 t)}r\cos t dt$$

$$= \int\sqrt{r^2(1-\sin^2 t)}r\cos t dt$$

$$= \int r\sqrt{(1-\sin^2 t)}r\cos t dt$$

**we know that $1 - \sin^2 t = \cos^2 t$**

$$= \int r^2 \cos t \cos t dt$$

$$= \int r^2 \cos^2 t dt$$

$r^2$ is constant and will go in front of integral and we will use formula: $\cos^2 t = \frac{1+\cos 2t}{2}$; then $\frac{1}{2}$ will ,also as a constant , go in front of integral…So:

$$= \frac{r^2}{2}\int(1+\cos 2t)dt$$

$$= \frac{r^2}{2}(t+\frac{1}{2}\sin 2t)$$

What happens to the borders of this integral?

Replacement was : $\begin{vmatrix} x = r\sin t \\ dx = r\cos t dt \end{vmatrix}$ , for x = - r is -r = r sin t , then sint = - 1 $\longrightarrow$ $t = -\frac{\pi}{2}$

for x = r is r = r sin t , sint = 1 $\longrightarrow$ $t = \frac{\pi}{2}$

**New boundaries are** $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

**Let's go back to the integral:**

$$V = \pi\int_a^b (y_1^2 - y_2^2)dx = \pi 4b\frac{r^2}{2}(t+\frac{1}{2}\sin 2t)\Bigg|_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= 2\pi br^2[(\frac{\pi}{2}+\frac{1}{2}\sin 2\frac{\pi}{2}) - (-\frac{\pi}{2}+\frac{1}{2}\sin(-2\frac{\pi}{2}))]$$

$$= 2\pi br^2\pi$$

$$= 2br^2\pi^2$$

So, after much effort, the finally solution is $V = 2br^2\pi^2$

**7. Calculate the length of the curve y = lnx from point x = $\sqrt{3}$ to point x = $\sqrt{8}$**

***Solution:***

Here we do not need a picture!

**Formula for calculating the length of the curve is: L=** $\int_a^b \sqrt{1+f`(x)^2}\,dx$ **, if we doing "by x"**

**y = ln x**

$y`=\frac{1}{x}$ So:

$$\int_a^b \sqrt{1+f`(x)^2}\,dx=\int_{\sqrt{3}}^{\sqrt{8}}\sqrt{1+(\frac{1}{x})^2}\,dx=\int_{\sqrt{3}}^{\sqrt{8}}\sqrt{1+\frac{1}{x^2}}\,dx=\int_{\sqrt{3}}^{\sqrt{8}}\sqrt{\frac{x^2+1}{x^2}}\,dx=\int_{\sqrt{3}}^{\sqrt{8}}\frac{\sqrt{x^2+1}}{x}dx = \text{replacement} =$$

$$= \left| \begin{matrix} x^2+1=t^2 \\ 2xdx=2tdt \\ xdx=tdt \\ dx=\frac{tdt}{x} \end{matrix} \right| \longrightarrow \left| \begin{matrix} x=\sqrt{3}\Rightarrow t=2 \\ x=\sqrt{8}\Rightarrow t=3 \end{matrix} \right|$$

$=\int_2^3 \frac{t}{x}\frac{tdt}{x}=\int_2^3 \frac{t^2dt}{x^2}=$ from the replacement is $x^2 = t^2 - 1$ ,so:

$=\int_2^3 \frac{t^2dt}{t^2-1}$ **[ +1 and – 1 as a "trick"]**

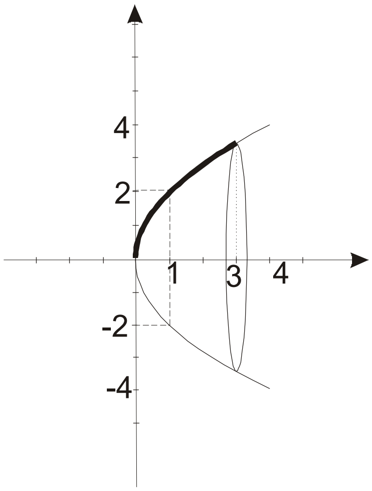
$$=\int_2^3 \frac{t^2-1+1}{t^2-1}dt=\int_2^3 (1+\frac{1}{t^2-1})dt= t+\ln\sqrt{\frac{t-1}{t+1}}\Bigg|_2^3 = (3+\ln\sqrt{\frac{3-1}{3+1}}) - (2+\ln\sqrt{\frac{2-1}{2+1}}) =$$

$$= 1+\sqrt{\ln\frac{3}{2}}$$

Solution is: $L = 1+\sqrt{\ln\frac{3}{2}}$

**8. Find area of body which make parable $y^2 = 4x$ rotating around x – line on a segment [0,3]**

***Solution:***




Formula for calculating this area is:

A= $2\pi \int_a^b f(x)\sqrt{1+f`(x)^2}\,dx$, by x $x \in [a,b]$

**Here are boundaries 0 and 3 ,obviously.**

**$y^2 = 4x$** from here is $y = 2\sqrt{x}$ $\longrightarrow$ $y` = 2\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$ $\longrightarrow$ $y`^2 = \frac{1}{x}$

$$A = 2\pi \int_a^b f(x)\sqrt{1+f`(x)^2}\,dx = 2\pi \int_0^3 2\sqrt{x}\sqrt{1+\frac{1}{x}}\,dx$$

$$= 2\pi \int_0^3 2\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}dx$$

$$= 4\pi \int_0^3 \sqrt{x+1}dx \quad \text{replacement}$$

$$= \begin{vmatrix} x+1 = t^2 \\ dx = 2tdt \end{vmatrix}$$

$$\int 2t^2 dt = 2\frac{t^3}{3} = \frac{2}{3}\sqrt{(x+1)^3}$$

$$= 4\pi \frac{2}{3}\sqrt{(x+1)^3}\Big|_0^3$$

$$= \frac{8\pi}{3}(8-1) = \frac{56\pi}{3}$$

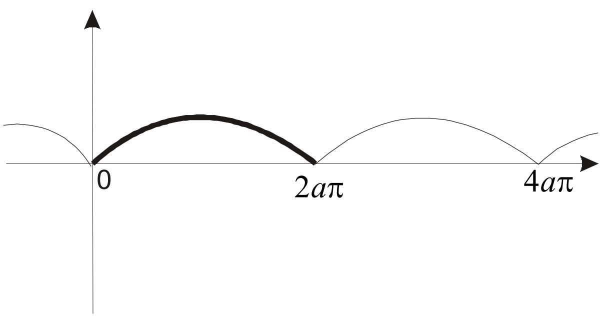
Solution is : A = $\frac{56\pi}{3}$

9. **Cycloid C is defined with parametric equations:** $x = a(t - \sin t)$ **and** $y = a(1 - \cos t)$

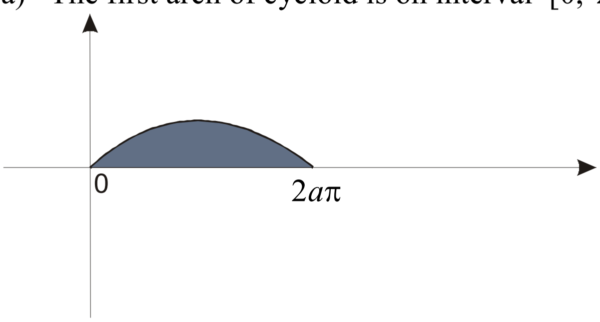
**Calculate:**

**a) area limited with one arch of cycloid**
**b) length of one arch**
**c) volume of body which caused one arch rotating around x – line**

***Solution***:




a) The first arch of cycloid is on interval $[0,\ 2a\pi]$




$$A=\int_a^b y\,dx$$

$$\left| \begin{matrix} y = 0 \Rightarrow a(1-\cos t) = 0 & t = 0 \\ y = 2a\pi \Rightarrow a(1-\cos t) = 2a\pi & t = 2\pi \end{matrix} \right|$$

$$x = a(t - \sin t) \longrightarrow dx = a(1-\cos t)dt$$

$$A=\int_a^b y\,dx = \int_0^{2\pi} a(1-\cos t)a(1-\cos t)dt = a^2\int_0^{2\pi}(1-\cos t)^2\,dt$$

$$\int (1-\cos t)^2\,dt = \int (1 - 2\cos t + \cos^2 t)dt = \int 1dt - 2\int \cos t dt + \int \frac{1+\cos 2t}{2}dt$$

$$= t - 2\sin t + \frac{1}{2}(t + \frac{1}{2}\sin 2t)$$

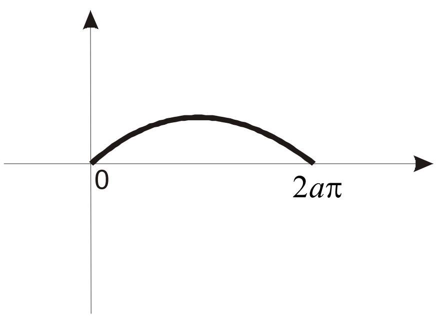
$$= t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t$$

$$= \frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t$$

$$A = a^2 \int_0^{2\pi} (1-\cos t)^2\, dt = a^2 \left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right)\Big|_0^{2\pi} = \mathbf{3}\, a^2 \pi$$

So: $\boxed{A = \mathbf{3}\, a^2 \pi}$

b)




$$L = \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t)}\, dt$$

$x = a(t - \sin t) \longrightarrow x' = a(1 - \cos t)$

$y = a(1 - \cos t) \longrightarrow y' = a\sin t$

$$x'^2 + y'^2 = [a(1-\cos t)]^2 + [a\sin t]^2 = a^2(1 - 2\cos t + \cos^2 t) + a^2\sin^2 t$$

$$= a^2(1 - 2\cos t + \cos^2 t + \sin^2 t)$$

$$= a^2(2 - 2\cos t)$$

$$= 2a^2(1 - \cos t)$$

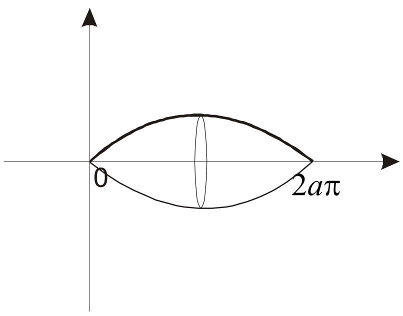
$$= 2a^2\, 2\sin^2\frac{t}{2}$$

$$= 4a^2\sin^2\frac{t}{2}$$

$$\int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t)}\, dt = \int_0^{2\pi} \sqrt{4a^2\sin^2\frac{t}{2}}\, dt = \int_0^{2\pi} 2a\sin\frac{t}{2}\, dt = 2a\int_0^{2\pi} \sin\frac{t}{2}\, dt = -4a\cos\frac{t}{2}\Big|_0^{2\pi} = 8a$$

$$\boxed{L = 8a}$$

c)




$$V = \pi \int_0^{2\pi} y^2 dx = \pi \int_0^{2\pi} [a(1-\cos t)]^2 a(1-\cos t)dt$$

$$= \pi \int_0^{2\pi} a^3 (1-\cos t)^3 dt$$

$$= a^3 \pi \int_0^{2\pi} (1-\cos t)^3 dt$$ here we must use $(a - b)^3$

$$= a^3 \pi \int_0^{2\pi} (1 - 3\cos t + 3\cos^2 t - \cos^3 t)dt$$

$$\int \cos^2 t = \int \frac{1+\cos 2t}{2} dt = \frac{1}{2}\int (1+\cos 2t)dt = \frac{1}{2}(t + \frac{1}{2}\sin 2t) = \frac{1}{2}t + \frac{1}{4}\sin 2t$$

$$\int \cos^3 tdt = \int \cos t \cos^2 tdt = \int \cos t(1-\sin^2 t) = \int \cos tdt - \int \cos t \sin^2 tdt = \left| \begin{matrix} \sin t = z \\ \cos tdt = dz \end{matrix} \right| = \sin t - \int z^2 dz =$$

$$\sin t - \frac{z^3}{3} = \sin t - \frac{\sin^3 t}{3}$$

**V =** $a^3 \pi [t - 3\sin t + 3( \frac{1}{2}t + \frac{1}{4}\sin 2t ) - (\sin t - \frac{\sin^3 t}{3})] \Big|_0^{2\pi}$ = simplify...= $5a^3 \pi^2$

**V=** $5a^3\pi^2$